NEW APPROACHES FOR THE POINT-FEATURE CARTOGRAPHIC LABEL PLACEMENT PROBLEM

Geraldo Regis Mauri
Federal University of Espírito Santo - UFES
Center for Agrarian Sciences, Alegre/ES
mauri@cca.ufes.br

Glaydston Mattos Ribeiro
Federal University of Espírito Santo - UFES
North University Center of Espírito Santo, São Mateus/ES
glaydstonribeiro@ceunes.ufes.br

Luiz Antonio Nogueira Lorena
National Institute for Space Research - INPE
Laboratory for Computing and Applied Mathematics, São José dos Campos/SP
lorena@lac.inpe.br

ABSTRACT

The point-feature cartographic label placement problem (PFCLP) consists in placing text labels adjacent to point features on a map. This paper proposes a 0-1 integer linear programming model for the PFCLP defined as the maximum number of free labels placed where all points must be labeled. We also present a Lagrangean decomposition technique based on graph partitioning where the PFCLP is represented by a conflict graph and partitioned into clusters. After the partitioning phase, some variables are copied to reduce de number of inter-clusters edges and the equality constraints associated to those variables copied are relaxed in a Lagrangean way. Computational experiments for sets of 25 instances with up to 1000 points showed that our Lagrangean decomposition provides good solutions better than CPLEX and the ones reported in the literature. We optimally solved all instances up to 750 points and the optimal was proven for 5 instances with 1000 points.

KEYWORDS. Label Placement. Linear Programming. Lagrangean Decomposition. Combinatorial Optimization.

1. Introduction

Label placement problems appear in several situations like in cartographic maps or in medical image analysis (Nascimento and Eades, 2008). They can be defined as follows: given a set of graphical features, such as points, we must place text labels adjacent to them identifying each one once at most. However, we can have, for each feature, an explicit enumerated list of candidate positions for the text label (discrete approach), or we can slide the text label around of each feature until finding the best position to place it (slider approach). In both cases, the text labels must be placed avoiding overlaps.

Basically, graphical features can be points, lines or polygons. For example, in a map, cities are represented by points, roadways by lines and states by polygons (areas). There are different approaches for each kind of feature.

The major problem in map labeling is that one with points known in the literature by point-feature cartographic label placement (PFCLP) (Klau and Mutzel, 2003). A good review about approaches for problems with lines or areas can be found at the “Map-Labeling Bibliography Web Site” (Wolff and Strijk, 1996), where we can see an illustrative chart of map labeling publications over the last 50 years.

In PFCLP the overlapping labels may be accepted or not. When overlaps are not accepted, we may attempt to either label a maximum number of points or determine the largest possible font size such that all points can be labeled. These problems are known as Label Number Maximization Problem (LNMP) and Label Size Maximization Problem (LSMP), respectively (Nascimento and Eades, 2008; Klau and Mutzel, 2003).

In the LNMP, some features cannot receive their labels. This problem is generally represented by a conflict graph where each node represents a candidate position for a label and each edge a potential conflict (overlap) between two candidate positions. Now, considering its objective (label number maximization), this problem can be seen as the traditional Maximum Independent Set Problem (MISP) (Strijk et al, 2000; Zoraster, 1990).

When overlaps are accepted, all points must be labeled and scaling is not allowed. Two problems are identified, the Maximum Number of Conflict Free Labels Problem (MNCFLP) and the Minimum Number of Conflicts Problem (MNCP). The MNCFLP (Ribeiro and Lorena, 2008b) is also known by Label Overlap Minimization Problem (Klau, 2002) and by Number of Labels Obstructed by at Least One Other Label (Christensen et al, 1995). The MNCP was recently presented by Ribeiro and Lorena (2008a,b) and their approach "spreads" the overlaps minimizing conflicts (edges) between candidate positions.

In this paper we concentrate on the MNCFLP with discrete positions, presenting a new 0-1 optimization model for it that to the best of our knowledge is the first model for this approach. Commercial solvers have difficulties for solving the MNCFLP large-scale instances available in the literature. So, we also present a Lagrangean decomposition that has generated good feasible solutions, outperforming recent results reported in the literature.

The remaining of the paper is organized as follows. Section 2 presents a brief review of PFCLP approaches. The proposed model and the Lagrangean decomposition are described in Sections 3 and 4, respectively. Computational experiments are reported in Section 5 and the conclusions are summarized in Section 6.

2. Literature review of the PFCLP with discrete candidate positions

The PFCLP is an optimization problem shown to be NP-hard (Formann and Wagner, 1991; Marks and Shieber, 1991). Exact approaches are limited to solve only small instances (Strijk et al, 2000; Zoraster, 1990), therefore heuristics and metaheuristics have been proposed.

Many approaches, exact or not, have their solution strategies based on conflict graphs. Let N be the number of points to be labeled and P_i a set of discrete positions for the label of point i (candidate positions). A conflict graph for PFCLP can be defined by $G=(V, E)$, where $V = \{v_{1,p_1^1}, v_{1,p_2^1}, \ldots, v_{1,p_{|P_1|}^1}, \ldots, v_{N,p_1^N}, v_{N,p_2^N}, \ldots, v_{N,p_{|P_N|}^N}\}$ is a set of nodes (all candidate positions) and $E = \{(v_{i,j}, v_{t,u}) : v_{i,j} \text{ and } v_{t,u} \in V, i \neq t\}$ a set of potential conflicts (overlaps) between candidate positions. For a good review about conflict graphs, see Atamtürk et al (2000).

For PFCLP with discrete candidate positions, Christensen et al (1995) have proposed a cartographic pattern (see Figure 1), where each position has a number to indicates a cartographic preference. In Figure 1, position 1 is the most suitable, i.e., the lower number indicates the best position. Starting from this pattern, the PFCLP can be defined as the problem of assigning the labels to one of its available candidate positions subject to conflict constraints and minimizing or maximizing an objective function.

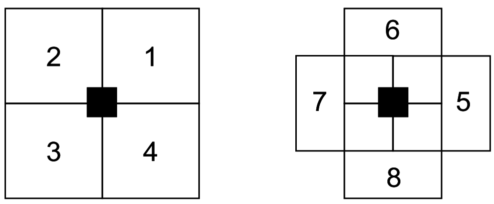


Figure 1: Cartographic pattern proposed by Christensen et al (1995).

Considering a problem with 2 points and 4 candidate positions for each one, we can get a conflict graph as presented in Figure 2 where dashed edges indicate conflicts between candidate positions of different points. The proportion of conflict free labels assesses the quality of the labeling (Strijk et al, 2000; Christensen et al 1995; Yamamoto et al, 2002). So, if the labels are placed in positions 1,1 and 2,1, this is a good labeling with all labels free.

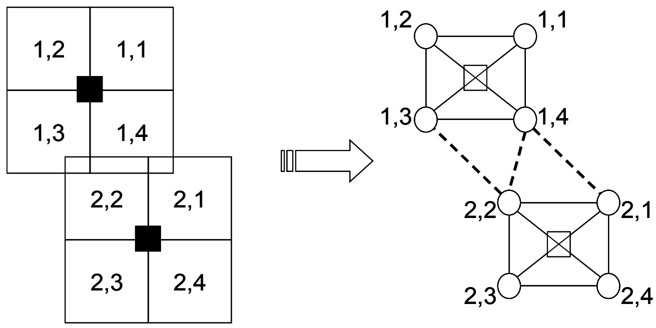


Figure 2: Example of a conflict graph (Ribeiro and Lorena, 2008b).

Considering the PFCLP as a Maximum Independent Set Problem (MISP), many researches are reported in the literature. But in the mathematical models field, Zoraster (1990) and Strijk et al (2000) have presented interesting contributions. Zoraster (1990) formulated mathematically the PFCLP working with conflict constraints and dummy candidate positions of high cost if the points could not be labeled. He also proposed a Lagrangean relaxation for the problem and obtained some computational results on small-scale instances. Strijk et al (2000) proposed mathematical formulations based on the so-called clique inequalities (Padberg, 1973), implementing a branch-and-cut algorithm, and testing several heuristics such as Tabu Search and Simulated Annealing. The authors used instances up to 950 points with 4 candidate positions.

Now if we look at the PFCLP as a MNCFLP, many researchers proposed heuristics and metaheuristics. Christensen et al (1995) presented a good review about the PFCLP and proposed a local search technique based on a discrete form of the gradient descent and a Simulated Annealing algorithm. Verner et al (1997) applied a Genetic Algorithm with mask such that if a label is in conflict, the changing of positions is allowed by crossover operators. Yamamoto et al (2002) proposed a Tabu Search algorithm while Yamamoto and Lorena (2005) developed a Constructive Genetic Algorithm and applied it to a set of large-scale instances.

Recently, Alvim and Taillard (2009) presented a POPMUSIC frame for the PFCLP. POPMUSIC (Partial Optimization Metaheuristic Under Special Intensification Conditions) was proposed by Taillard and Voss (2001) and its basic idea consists in locally optimizing sub-parts of a solution, once a solution of the problem is available. The local optimizations are repeated until no improvements are found. For the local optimizations, the authors implemented a new version of the Tabu Search proposed by Yamamoto et al (2002).

Alvim and Taillard (2009) have applied POPMUSIC to instances proposed in the literature by Yamamoto et al (2002) and to real instances obtained from Switzerland road network. The POPMUSIC have presented good solutions, better than other approaches, in small computational times.

Finally, looking PFCLP as MNCP point of view, Ribeiro and Lorena (2006, 2008b) introduced this approach to minimize the number of conflicts (edges in the remaining conflict graph). The authors have proposed two 0-1 optimization models and a Lagrangean heuristic. Regarding the optimization models, the one proposed in Ribeiro and Lorena (2008b) has a smaller number of constraints.

Considering that the conflict graph can be large and that it can become hard to deal with it, Wagner et al (2001) presented an approach to reduce it. They proposed three rules to reduce the graph size without altering the set of optimal solutions. For the MNCFLP, the following rules are applicable:

- Rule 1: If a point p has a candidate position p_i without any conflicts, declare p_i to be part of the solution, and eliminate all other candidate positions of p;

- Rule 2: If a point p has a candidate position p_i that is only in conflict with a candidate position q_k, and q has a candidate position q_j ($j \neq k$) that is only overlapped by candidate position p_l ($l \neq i$), then add p_i and q_j to the solution and eliminate all other candidate positions of p and q;

- Rule 3: If p has only one candidate position p_i left, and the candidate positions overlapping p_i form a clique, then declare p_i to be part of the solution and eliminate all candidate positions that overlap p_i.

These rules are applied exhaustively. After eliminating a candidate p_i, we must check recursively whether the rules can be applied in the neighborhood of p_i.

For more details and algorithms, see the "Map-Labeling Bibliography Web Site" at http://i11www.iti.uni-karlsruhe.de/~awolff/map-labeling/bibliography/.

3. The proposed model

In this section we propose a 0-1 integer linear programming model for the PFCLP as MNCFLP, i.e., fixed label sizes, discrete positions for the labels, all points must be labeled and we are looking for the maximum number of conflict free labels.

Let $x_{i,j}$ be a binary variable to represent the candidate position j of the point i for all $i \in \{1, ..., N\}$ and $j \in P_i$. If $x_{i,j} = 1$ the label of point i must be placed at position j, and $x_{i,j} = 0$ otherwise. For each candidate position of point i is associated a profit represented by $w_{i,j}$. Now let $S_{i,j}$ be a set of pairs $(t,u): t \neq i$ composed of candidate positions $x_{t,u}$ that present potential conflicts with $x_{i,j}$. Thus, the MNCFLP 0-1 optimization model is:

MNCFLP: *v(MNCFLP)* = **Maximize:**

$$\sum_{i=1}^{N} \sum_{j \in P_i} w_{i,j} x_{i,j} - \sum_{i=1}^{N} z_i \tag{1}$$

Subject to:

$$\sum_{j \in P_i} x_{i,j} = 1 \qquad \forall i = 1,...,N \tag{2}$$

$$x_{i,j} + x_{t,u} - z_i \leq 1 \qquad \forall i = 1,...,N; \forall j \in P_i; i \neq t; (t,u) \in S_{i,j} \tag{3}$$

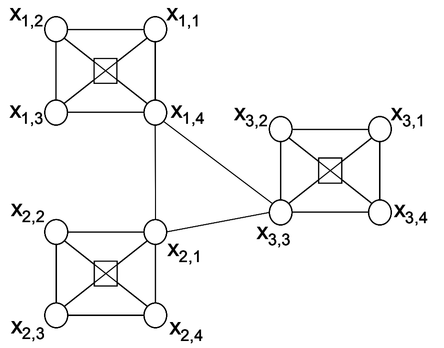
$$x_{i,j}, x_{t,u}, z_i \in \{0,1\} \qquad \forall i = 1,...,N; \forall j \in P_i; (t,u) \in S_{i,j} \tag{4}$$

When the binary variable $z_i = 1$, it means that some candidate position is overlapping point i, $z_i = 0$, otherwise. Constraints (2) ensure that each point i must be labeled, i.e., some candidate position $x_{i,j}$ must be equal to 1. Constraints (3) ensure the correct assignment to variables z when overlaps (conflicts) are inevitable and constraints (4) impose binary variables. Constraints (2) can also be seen as "conflicts" but between candidate positions of the same point.

The overlapping variables z_i appear subtracting in the objective function to be maximized which should seek for solutions with zero value for these variables.

Model (1)-(4) is similar to the one proposed by Zoraster (1990) and Ribeiro and Lorena (2008a) but it allows allocating all labels maximizing the number of conflict free labels. Figure 3(a) presents a conflict graph with respective constraints (3) in Figure 3(b).

Thus, if $x_{1,4} = x_{2,1} = 1$ the label of point 1 is overlapped by the label of point 2, consequently point 1, represented by z_1, is overlapped by point 2; and point 2, represented by z_2, is overlapped by point 1. Finally, $z_1 = z_2 = 1$ (ensured by constraint 3 presented in Figure 3(b)) therefore indicating two overlapped labels or "overlapped points".



$$\begin{array}{l} x_{1,4} + x_{2,1} - z_1 \le 1 \\ x_{1,4} + x_{3,3} - z_1 \le 1 \\ x_{2,1} + x_{1,4} - z_2 \le 1 \\ x_{2,1} + x_{3,3} - z_2 \le 1 \\ x_{3,3} + x_{1,4} - z_3 \le 1 \\ x_{3,3} + x_{2,1} - z_3 \le 1 \end{array}$$

(a) **(b)**

Figure 3: Conflict graph and conflict constraints for the MNCFLP model.

The objective function maximizes the number of conflict free labels and the sum of the variables $x_{i,j}$ (Equation 1), ignoring the cartographic preferences $w_{i,j}$, will be always equal to the total number of points (ensured by Constraints 2). The number of overlapped labels will be given by the sum of variables z. Therefore, ignoring the cartographic preferences (considering all $w_{i,j}$ = 1, for example), the value for the objective function will be exactly the number of conflict free labels.

4. Lagrangean decomposition

The Lagrangean decomposition is a special case of Lagrangean relaxation that consists of partitioning the original problem into several sub-problems creating a copy of the decision variables in each one of the generated sub-problems. These "clones" are used in the sub-problems' constraints and new constraints ensure the equality between them and the original variables. Thus, the Lagrangean decomposition appears when we relax in a Lagrangean way these new constraints (Chardaire and Sutter, 1995; Guignard, 2003). Therefore, it is important to copy variables as little as possible to reduce the number of new constraints.

For the MNCFLP model (1)-(4), the conflict graph G will be partitioned into m ($m \le N$) clusters of vertices with $V = V_1 \cup V_2 \cup ... \cup V_m$, and $V_i \cap V_j = \emptyset$, $\forall\ i,j \in \{1,\ldots,m\}$ forming sub-graphs $G_k = (V_k, E_k)\ \forall\ k = 1,...,m$. Now, let $X_k = V - V_k$ be a set of the vertices not included in cluster k and C_k be the set of copied variables on cluster k. Figure 4 below is used to describe our Lagrangean decomposition.

To copy variables as little as possible, the graph partitioning phase is a challenge. But for the conflict graph provided by *MNCFLP*, we can use the technique of vertices contraction which consists on grouping all candidate positions of the same point i forming a single vertex (see Figure 4(b)). In Figure 4 the squares and circles indicate the points to be labeled and the candidate positions, respectively.

The contraction of vertices generates a conflict graph G between points and not between candidate positions. Therefore, the graph partitioning between points generates sub-problems (Figure 4(c)) where conflicts between candidate positions of the same point are maintained (constraint 2) resulting in a stronger relaxation for the PFCLP.

After the partitioning of G, the contractions are expanded (Figure 4(d)) resulting in the original conflict graph with the inter-cluster edges (dashed edges on Figure 4(d)) and the sub-problems (clusters 1 and 2).

Now, we must determine which binary variables (vertices) must be copied. A good strategy is proposed by Sachdeva (2004) that copies vertices with the greatest number of inter-clusters edges. After copying some vertex, the approach redefines the number of inter-clusters edges and a new vertex is selected to be copied. This process is repeated until all the necessary copies are carried out, i.e., all the edges inter-clusters are eliminated.

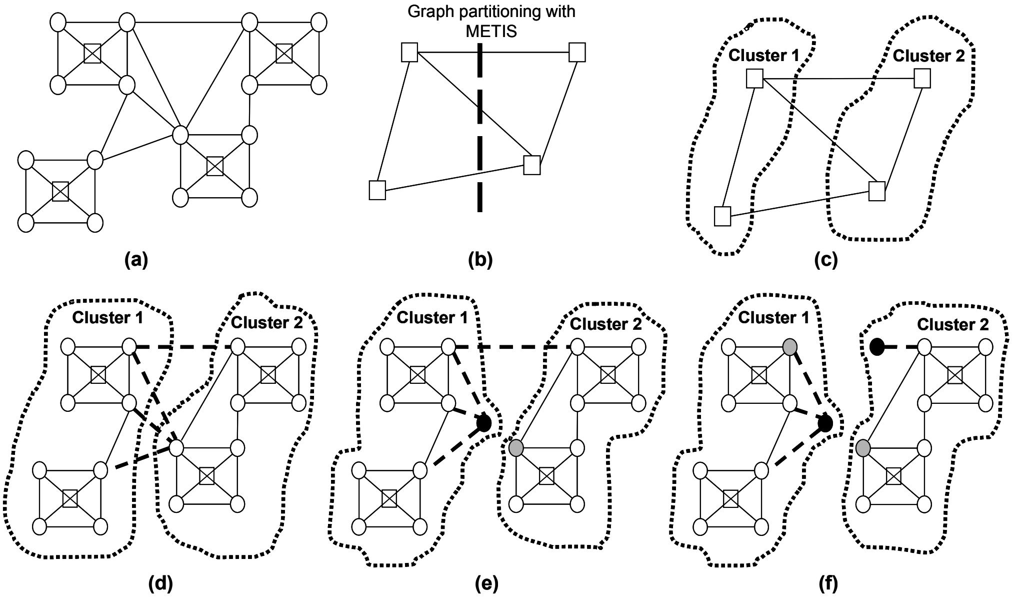


Figure 4: Graph decomposition for the *PFCLP*.

As shown in Figure 4(e), the first vertex copied presents three inter-clusters edges (black vertex is a copy of the gray one). The simple copy of the gray vertex to cluster 1 removes three inter-cluster edges. The process is repeated and finished in Figure 4(f), leaving two independent clusters. Now ensuring the equality between copies (black vertices) and original (gray vertices) variables, the problem is decomposed into two clusters. Thus, the *MNCFLP* decomposition into m ($m \leq N$) clusters can be described as follows:

MNCFLP^m: *v(MNCFLP*^{m}*)* = **Maximize:**

$$\sum_{k=1}^{m}\left(\sum_{(i,j)\in V_k} w_{i,j}x_{i,j} - \sum_{(i,j)\in V_k} z_i\right) \tag{5}$$

Subject to:

$$\sum_{j\in P_i} x_{i,j} = 1 \qquad \forall\, i\in V_k; k = 1,...,m \tag{6}$$

$$x_{i,j} + x_{t,u} - z_i \leq 1 \qquad \forall\,(i,j)\in V_k; i\neq t; (t,u)\in S_{i,j}; k = 1,...,m \tag{7}$$

$$x_{i,j} + x^k_{t,u} - z_i \leq 1 \qquad \forall\,(i,j)\in V_k; (t,u)\in C_k \cap S_{i,j}; k = 1,...,m \tag{8}$$

$$x_{i,j} + x^k_{t,u} - z^k_t \leq 1 \qquad \forall\,(i,j)\in V_k; (t,u)\in C_k \cap S_{i,j}; k = 1,...,m \tag{9}$$

$$x_{t,u} = x^k_{t,u} \qquad \forall\,(t,u)\in C_k \cap X_k; k = 1,...,m \tag{10}$$

$$z_t = z^k_t \qquad \forall\,(t,u)\in C_k \cap X_k; k = 1,...,m \tag{11}$$

$$x_{i,j}, x_{t,u}, z_i, z_t, x^k_{t,u}, z^k_t \in \{0,1\} \qquad \forall\,(i,j)\in V_k; (t,u)\in V_k \cup C_k; k = 1,...,m \tag{12}$$

Variables $x^k_{t,u}$ and z^k_t represent, respectively, the copies of $x_{t,u}$ and z_t into cluster k. The copies are not considered in the objective function, i.e., their coefficients are zero. Constraints (6) and (7) deal only with the edges whose vertices are internal to each cluster (sub-problem) k. Constraints (8) and (9) indicate the edges whose vertices are in different clusters (inter-clusters edges) and constraints (10) and (11) ensure the equality between original and copy variables.

Relaxing constraints (10) and (11) in a Lagrangean way with vectors of unrestricted Lagrangean multipliers **α** and **β,** the problem $MNCFLP^m$ (indirectly the *MNCFLP*) can be divided into *m* independent sub-problems. Each sub-problem *k* can be defined as follows:

$LD_{\alpha\beta}MNCFLP_k$: $v(LD_{\alpha\beta}MNCFLP_k)$ = **Maximize:**

$$\sum_{(i,j)\in V_k}\left(w_{i,j}-\sum_{d\neq k}\alpha_{i,j}^{d}\right)x_{i,j}+\sum_{(t,u)\in C_k}\left(\alpha_{t,u}^{k}\right)x_{t,u}^{k}-\sum_{(i,j)\in V_k}\left(z_i-\sum_{d\neq k}\beta_i^{d}\right)z_i+\sum_{(t,u)\in C_k}\left(\beta_t^{k}\right)z_t^{k} \quad (13)$$

Subject to:

$$\sum_{j\in P_i} x_{i,j}=1 \qquad \forall\, i\in V_k \quad (14)$$

$$x_{i,j}+x_{t,u}-z_i\leq 1 \qquad \forall\,(i,j)\in V_k;\, i\neq t;(t,u)\in S_{i,j} \quad (15)$$

$$x_{i,j}+x_{t,u}^{k}-z_i\leq 1 \qquad \forall\,(i,j)\in V_k;(t,u)\in C_k\cap S_{i,j} \quad (16)$$

$$x_{i,j}+x_{t,u}^{k}-z_t^{k}\leq 1 \qquad \forall\,(i,j)\in V_k;(t,u)\in C_k\cap S_{i,j} \quad (17)$$

$$x_{i,j},x_{t,u},z_i,x_{t,u}^{k},z_t^{k}\in\{0,1\} \qquad \forall\,(i,j)\in V_k;(t,u)\in V_k\cup C_k \quad (18)$$

Finally, the *MNCFLP* relaxation in *m* sub-problems is given by expression (19) and the corresponding Lagrangean dual is presented in expression (20).

$$LD_{\alpha\beta}MNCFLP^m:\quad v(LD_{\alpha\beta}MNCFLP^m)=\sum_{k=1}^{m} v(LD_{\alpha\beta}\,MNCFLP_k) \quad (19)$$

$$DLD_{\alpha\beta}MNCFLP^m:\ v(DLD_{\alpha\beta}MNCFLP^m)=\underset{\alpha,\beta\ unrestricted}{\mathbf{Min}}\{v(LD_{\alpha\beta}\,MNCFLP^m)\} \quad (20)$$

Now, we have *m* small and independent sub-problems that can be solved by a commercial solver. Consequently, a subgradient algorithm can be implemented to manage these sub-problems and to update the multipliers and the step size.

For our computational tests, we have used CPLEX 10.0.1 (Ilog, 2006) for solving the sub-problems and the heuristic METIS (Karypis and Kumar, 1998) for graph partitioning task that, according to Warrier et al (2005), presents good results on minimizing the number of edges with endings in different clusters.

We have implemented a subgradient algorithm to solve the Lagrangean dual (20) based on the one proposed by Narciso and Lorena (1999). Our algorithm is similar to the one proposed by Held and Karp (1970) and it updates the Lagrangean multipliers considering step sizes based on the relaxed solutions and the feasible solutions obtained with a Lagrangean heuristic (Figure 5).

The heuristic shown in Figure 5 is a greedy heuristic that verifies the better changing of candidate positions to generate an improved feasible solution. Firstly (lines 6 to 10) the solution obtained by the relaxation in *m* sub-problems is mounted. The candidate position selected for each point *i* is verified on lines 15 to 19 and changed for all the other ones (lines 20 to 34). A new candidate position is stored if a better solution is obtained, and otherwise the change is discarded (lines 31 and 32). This procedure is repeated for all the points (lines 14 to 35). The whole process is repeated when a better solution is found (lines 12 to 40).

5. Computational experiments

Several computational experiments were performed to evaluate our mathematical model and our Lagrangean decomposition, considering a set of randomly generated instances with 100, 250, 500, 750 and 1000 points proposed by Yamamoto et al (2002). These instances have been used in several works about PFCLP (see Section 2) and are available at http://www.lac.inpe.br/~lorena/instancias.html. The Lagrangean decomposition was implemented in C++ and the experiments performed on a PC with Pentium Dual Core of 1.73 GHz with 1GB of RAM memory.

```
LAGRANGEAN HEURISTIC
x_{i,j}*: value for x_{i,j} on integer solution (viable);
x_{i,j}: value for x_{i,j} on sub-problem for vertex (i,j);
p: selected candidate position;
r: candidate position selected before the best
change;
q: candidate position selected after the best change;
FOR i ← 1 TO N DO
FOR all j ∈ P_i DO
x_{i,j}* ← x_{i,j};
END-FOR;
      END-FOR;
      fo* ← COMPUTE (the objective function value);
      DO
improve ← false;
FOR i ← 1 TO N DO
   FOR all j ∈ P_i DO
      IF (x_{i,j}* = 1) THEN
         p ← j;
      END-IF;
   END-FOR;
   FOR all j ∈ P_i DO
      IF (j ≠ p) THEN
         x_{i,p} ← 0;
         x_{i,j} ← 1;
         fo = COMPUTE (the objective function value);
         IF (fo > fo*) THEN
            r ← p;
            q ← j;
            fo* ← fo;
            improve ← true;
         END-IF;
         x_{i,p} ← 1;
         x_{i,j} ← 0;
      END-IF;
   END-FOR;
END-FOR;
IF (improve) THEN
   x_{i,r}* ← 0;
   x_{i,q}* ← 1;
END-IF;
      WHILE (improve);
```

Figure 5: Lagrangean heuristic for the *MNCFLP*.

As considered in Zoraster (1990), Christensen et al (1995), Ribeiro and Lorena (2008a,b), Verner et al (1997), Yamamoto and Lorena (2005), Alvim and Taillard (2009) and others, the cartographic preferences were not considered ($w_{i,j} = 1$) for all candidate positions, and the total number of candidate positions was equal to 4 ($P_i = \{1,2,3,4\}\ \forall i = 1,\ldots,N$). The reduction heuristic proposed by Wagner et al (2001) was used to reduce the initial conflict graph, i.e., the conflict graph (problem) was reduced before we apply the METIS heuristic. However let us mention that Rule 3 is not applied in our computational experiments.

Tables 1 and 2 present the results for the instances with 750 and 1000 points, respectively. Both CPLEX and our Lagrangean decomposition presented optimal solutions in a computational time inferior to 3 seconds for the instances with 100, 250, and 500 points (99.68% of conflict free labels for the instances with 500 points and 100% for 100 and 250 points).

In Tables 1 and 2, m indicates the number of clusters used; LB and UB indicate the upper and lower bounds found, respectively; Gap indicates the difference between LB and UB ($Gap = \frac{(UB - LB)}{LB} \times 100$); Time shows the total processing time and the last line presents the arithmetic average to each column; $\overline{N}$ is the number of points obtained after the reduction heuristic proposed by Wagner et al (2001); The number m was estimated based on the ones reported by Ribeiro and Lorena (2008a).

Table 1: Results for instances with 750 points.

Inst.	$\overline{N}$	$DLD_{\alpha\beta}MNCFLPC^m$					CPLEX			
		m	LB	UB	Gap	Time (s)	LB	UB	Gap	Time (s)
1	425	10	739*	739	0	1.10	739	741.83	0.38	44.67
2	427	10	736*	736	0	1.71	735	744.46	1.29	2033.56
3	395	10	731*	731	0	32.18	730	743.55	1.86	8957.94
4	393	10	741*	741	0	1.66	741	742.00	0.13	10.69
5	370	10	739*	739	0	1.38	739	743.78	0.65	143.73
6	407	10	730*	730	0	22.82	730	743.50	1.85	15210.70
7	421	10	737*	737	0	2.44	737	742.00	0.68	70.05
8	414	10	736*	736	0	1.49	736	743.02	0.95	1913.95
9	395	10	726*	726	0	8.62	724	744.67	2.85	5183.67
10	382	10	743*	743	0	40.23	743*	743.00	0	2.13
11	395	10	733*	733	0	2.65	732	744.00	1.64	2992.88
12	382	10	734*	734	0	1.87	733	742.50	1.30	20.08
13	373	10	743*	743	0	2.08	743*	743.00	0	1.94
14	398	10	728*	728	0	5.15	728	743.58	2.14	8911.52
15	406	10	730*	730	0	32.56	728	744.25	2.23	15095.80
16	401	10	729*	729	0	396.10	729	744.00	2.06	11111.20
17	367	10	729*	729	0	6.77	728	744.62	2.28	10142.30
18	407	10	737*	737	0	1.69	737	742.67	0.77	154.44
19	407	10	740*	740	0	3.59	740	742.50	0.34	9.97
20	422	10	737*	737	0	2.91	737	742.67	0.77	81.36
21	447	10	731*	731	0	19.91	731	744.67	1.87	7324.94
22	378	10	744*	744	0	0.91	744*	744.00	0	3.00
23	422	10	731*	731	0	3.90	731	742.00	1.50	36.77
24	422	10	732*	732	0	8.33	732	744.40	1.69	5569.06
25	398	10	732*	732	0	5.73	732	743.33	1.55	13494.50
Avg.	**402.16**	**10**	**734.72**	**734.72**	**0**	**24.31**	**734.36**	**743.36**	**1.23**	**7139.64**

* The optimal was proven.

Table 1 presents the results obtained for instances with 750 points. Our Lagrangean decomposition found the optimal solutions for all the instances with an average computational time of 24.31 seconds. CPLEX presented an average residual gap of 1.23% proving optimality for only 3 of the 25 instances (see instances with asterisk) with an average time of 7139.64 seconds. Our Lagrangean decomposition found 97.96% of conflict free labels while CPLEX found 97.91% (Equation 21).

$$\text{Conflicts free labels}(\%) = \frac{LB}{N} \times 100 \quad (21)$$

The results obtained for the instances with 1000 points are reported in Table 2. Our Lagrangean decomposition has proved optimality for 5 of the 25 instances with an average residual gap of 0.48%. CPLEX cannot prove the optimality for any instance and has presented a gap of 10.64%. Our Lagrangean decomposition times are in magnitude similar to the ones provided by CPLEX. Our Lagrangean decomposition provided 93.74% of conflict free labels while CPLEX found 90.13%.

Table 2: Results for instances with 1000 points.

Inst.	$\overline{N}$	$DLD_{\alpha\beta}MNCFLPC^{m}$					CPLEX			
		m	LB	UB	Gap	Time (s)	LB	UB	Gap	Time (s)
1	757	25	939*	939.98	0.10	1348.90	918	995.27	8.42	8234.54
2	742	25	933	939.62	0.71	14407.07	892	996.56	11.72	7944.66
3	720	20	934*	934.00	0	2751.70	911	994.81	9.20	8010.25
4	749	40	929	939.90	1.17	14456.49	882	996.94	13.03	7675.42
5	732	15	960	962.33	0.24	14414.26	945	997.00	5.50	7585.55
6	702	20	932*	932.92	0.10	1400.71	920	995.42	8.20	8165.56
7	729	25	928	931.99	0.43	14419.30	897	996.92	11.14	8102.22
8	731	20	940	942.21	0.24	14419.89	908	998.33	9.95	8037.76
9	701	20	923	930.32	0.79	15144.10	893	996.08	11.54	8074.16
10	733	20	943	946.48	0.37	14704.62	901	997.14	10.67	8220.80
11	744	25	947*	947.90	0.10	3334.22	938	995.41	6.12	7889.20
12	728	25	934	936.44	0.26	14564.28	903	996.22	10.32	7858.28
13	680	25	954	956.03	0.21	14529.87	928	997.00	7.44	7549.22
14	743	25	930	941.15	1.20	14529.70	897	996.15	11.05	8154.78
15	756	25	932	937.00	0.54	16453.17	904	994.85	10.05	8148.46
16	707	25	928	936.01	0.86	14452.63	876	997.17	13.83	8223.61
17	671	25	937	939.63	0.28	14467.97	929	996.75	7.29	8105.20
18	734	25	946	947.27	0.13	14509.09	923	995.75	7.88	7927.24
19	708	25	950	953.25	0.34	15264.32	842	996.83	18.39	8724.84
20	724	25	929	945.84	1.81	14419.34	909	997.00	9.68	7910.86
21	749	25	928	932.22	0.45	14436.28	842	997.75	18.50	8219.96
22	694	25	952	955.41	0.36	14406.28	928	997.25	7.46	8141.83
23	723	25	933	940.23	0.77	14611.45	853	995.39	16.69	8516.06
24	722	25	929	934.39	0.58	14463.91	877	995.25	13.48	8664.67
25	712	25	945*	945.70	0.07	3513.24	917	995.50	8.56	8181.32
Avg.	**723.64**	**24.20**	**937.40**	**941.93**	**0.48**	**12216.91**	**901.32**	**996.35**	**10.64**	**8090.66**

* The optimal was proven.

Table 3 shows a comparison among the conflict free labels proportion obtained by a set of the main works about the MNCFLP. We can observe that our Lagrangean decomposition presents better results proving the optimality for several instances. We also note a slight improvement among the methods found in the literature detaching the improvements presented by our Lagrangean decomposition.

The average computational times were not reported on Table 3 due to the diversity of the computers used. However, we can note that our computational times (see Tables 1 and 2) are higher. This fact is acceptable because we consider exact methods to solve the MNCFLP.

6. Conclusions

In the literature, we can note a tough dispute for the best solutions to the Point-Feature Cartographic Label Placement Problems (see Section 2). However, most of the methods are based on heuristics and metaheuristics that do not allow us to verify how near the solution is from the optimal one, mainly when PFCLP is modeled according to the maximum number of conflict free labels approach.

This paper has presented a new 0-1 integer linear programming model to the MNCFLP. A method based on Lagrangean decomposition presented tight gaps for a set of instances up to 1000 points. The decomposition method has proved the optimality for several instances of the literature for the first time, with gaps smaller than the ones found by CPLEX. In addition, we believe that our Lagrangean decomposition is an interesting tool for solving problems represented by conflict graphs.

Table 3: Comparison with the literature: proportion (%) of conflict free labels.

Methods	Points			
	250	**500**	**750**	**1000**
$DLD_{\alpha\beta}MNCFLPC^{m}$	**100.00***	**99.68***	**97.96***	**93.74**
CPLEX with the proposed model *MNCFLP*	**100.00***	**99.68***	97.91	90.13
Pop(asc) (Alvim and Taillard, 2009)	**100.00**	99.67	97.72	92.68
Pop(10) (Alvim and Taillard, 2009)	**100.00**	99.67	97.46	91.94
Pop(30) (Alvim and Taillard, 2009)	**100.00**	99.67	97.72	92.54
Pop(70) (Alvim and Taillard, 2009)	**100.00**	99.67	97.73	92.58
Tabu(50n) (Alvim and Taillard, 2009)	**100.00**	99.57	97.53	91.54
Tabu(100n) (Alvim and Taillard, 2009)	**100.00**	99.57	97.54	91.54
Tabu(500n) (Alvim and Taillard, 2009)	**100.00**	99.57	97.55	91.59
Column Generation (Ribeiro and Lorena, 2008b)	**100.00**	99.67	97.67	92.40
LagClus (Ribeiro and Lorena, 2008b)	**100.00**	99.67	97.65	91.42
GRASP(6) (Cravo et al, 2008)	**100.00**	99.67	97.72	92.20
GRASP(5) (Cravo et al, 2008)	**100.00**	99.67	97.70	92.02
CGA (Yamamoto and Lorena, 2005)	**100.00**	99.60	97.10	90.70
Tabu Seach (Yamamoto et al, 2002)	**100.00**	99.26	96.76	90.00
GA with masking (Verner et al, 1997)	99.98	98.79	95.99	88.96
GA (Verner et al, 1997)	98.40	92.59	82.38	65.70
Simulated Annealing (Christensen et al, 1995)	99.90	98.30	92.30	82.09
3-opt Gradient Descent (Christensen et al, 1995)	99.76	97.34	89.44	77.83
2-opt Gradient Descent (Christensen et al, 1995)	99.36	95.62	85.60	73.37
Gradient Descent (Christensen et al, 1995)	95.47	86.46	72.40	58.29
Greedy Algorithm (Christensen et al, 1995)	88.82	75.15	58.57	43.41

* The optimal was proven.

Acknowledgments

The authors acknowledge the useful comments and suggestions provided by the anonymous referee, and FAPESP (process 04/11053-9) and CNPq (processes 305255/2006-5 and 471837/2008-3) for the partial financial support given to the development of this work.

References

Alvim, A. C. F. and Taillard, E. D. (2009), POPMUSIC for the point feature label placement, *European Journal of Operational Research*, 192(2), 396-413.

Atamtürk, A., Nemhauser, G. L. and Savelsbergh, M. W. P. (2000), Conflict graphs in solving integer programming problems, *European Journal of Operational Research*, 121(1), 40-55.

Chardaire, P. and Sutter, A. (1995), A decomposition method for quadratic zero-one programming, *Management Science*, 41(4), 704-712.

Christensen, J., Marks, J. and Shieber, S. (1995), An empirical study of algorithms for point-feature label placement, *ACM Transactions on Graphics*, 14(3), 203-232.

Cravo, G. L., Ribeiro, G. M. and Lorena, L. A. N. (2008), A greedy randomized adaptive search procedure for the point-feature cartographic label placement, *Computers & GeoSciences*, 34(4), 373-386.

Formann, M. and Wagner, F. (1991), A packing problem with applications to lettering of maps. *Proceedings of the Seventh Annual ACM Symposium on Computational Geometry*, New Hampshire, 281-288.

Guignard, M. (2003), Lagrangean relaxation, *TOP*, 11(2), 151-200.

Held, M. and Karp, R. M. (1970), The traveling salesman problem and minimum spanning trees, *Operations Research*, 18(6), 1138-1162.

ILOG, *Ilog CPLEX 10.0 user's manual*, France, 2006.

Karypis, G. and Kumar, V. (1998), Multilevel k-way partitioning scheme for irregular graphs, *Journal of Parallel and Distributed Computing*, 48(1), 96-129.

Klau, G. W., A combinatorial approach to orthogonal placement problems, *Thesis,* Saarbrueck University, Germany, 2002.

Klau, G. W. and Mutzel, P. (2003), Optimal labeling of point features in rectangular labeling models, *Mathematical Programming*, 94(2-3), 435-458.

Marks, J. and Shieber, S., The computational complexity of cartographic label placement, *Technical Report TR-05-91, Advanced Research in Computing Technology,* Harvard University, USA, 1991.

Narciso, M. G. and Lorena, L. A. N. (1999), Lagrangean/surrogate relaxation for generalized assignment problems, *European Journal of Operational Research*, 114(1), 165-177.

Nascimento, H. A. D. and Eades, P. (2008), User hints for map labeling, *Journal of Visual Languages & Computing*, 19(1), 39-74.

Padberg, M. W. (1973), On the facial structure of set packing polyhedra, *Mathematical Programming*, 5, 199–215.

Ribeiro, G. M. and Lorena, L. A. N. (2006), Heuristics for cartographic label placement problems, *Computers & GeoSciences*, 32(6), 739-748.

Ribeiro, G. M. and Lorena, L. A. N. (2008a), Column generation approach for the point-feature cartographic label placement problem, *Journal of Combinatorial Optimization*, 15(2), 147-164.

Ribeiro, G. M. and Lorena, L. A. N. (2008b), Lagrangean relaxation with clusters for point-feature cartographic label placement problems, *Computers & Operations Research*, 35(7), 2129-2140.

Sachdeva, S., Development of a branch and price approach involving vertex cloning to solve the maximum weighted independent set problem, *Dissertation,* A&M University, USA, 2004.

Strijk, T., Verweij, B. and Aardal, K., Algorithms for maximum independent set applied to map labeling, *Technical Report,* Universiteit Utrecht, Nederland, 2000.

Taillard, E. and Voss, S., POPMUSIC: Partial Optimization Metaheuristic Under Special Intensification Conditions, In: Ribeiro, C. and Hansen, P. (Eds.), *Essays and Survey in Metaheuristics*. Kluwer Academic Publishers, Boston, 613-629, 2001.

Verner, O. V., Wainwright, R. L. and Schoenefeld, D. A. (1997), Placing text labels on maps and diagrams using genetic algorithms with masking, *Journal on Computing*, 9(3), 266-275.

Wagner, F., Wolff, A., Kapoor, V. and Strijk, T. (2001), Three rules suffice for good label placement, *Algorithmica*, 30(2), 334-349.

Warrier, D., Wilhelm, W. E., Warren, J. S. and Hicks, I. V. (2005), A branch-and-price approach for the maximum weight independent set problem, *Networks*, 46(4), 198-209.

Wolff, A. and Strijk, T. (1996), *The map labeling bibliography*. Available at: <http://i11www.ilkd.uni-karlsruhe.de/~awolff/map-labeling/bibliography/>.

Yamamoto, M. and Lorena, L. A. N., A constructive genetic approach to point-feature cartographic label placement, in: Ibaraki, T., Nonobe, K. and Yagiura, M. (Eds.), *Metaheuristics: progress as real problem solvers*, Springer, Berlin, 285-300, 2005.

Yamamoto, M., Câmara, G. and Lorena, L. A. N. (2002), Tabu search heuristic for point-feature cartographic label placement, *GeoInformatica*, 6(1), 77-90.

Zoraster, S. (1990), The solution of large 0-1 integer programming problems encountered in automated cartography, *Operations Research*, 38(5), 752-759.